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Tight points and countable fan-tightness

Angelo Bella^a, Jan van Mill^{b,*}

^a *Dipartimento di Matematica, Città universitaria, viale A. Doria 6, 95125 Catania, Italy*

^b *Department of Mathematics and Computer Science, Vrije Universiteit, De Boelelaan 1081a, 1081 HV Amsterdam, Netherlands*

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Abstract

The countable spaces whose product with the sequential fan $S_\mathfrak{c}$ have countable tightness are characterized. As a consequence, it is shown that if $X \times S_\mathfrak{c}$ has countable tightness then X has countable fan-tightness.

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1. Introduction

The aim of this paper is to characterize the countable spaces whose product with the sequential fan $S_\mathfrak{c}$ have countable tightness. Our results answer questions posed by Arhangel'skiĭ and Bella [2].

Henceforth all spaces are assumed at least T_1 . ω denotes the set (or the discrete space) of all the integers as well as the first infinite ordinal. \mathfrak{c} denotes 2^ω .

$S_\mathfrak{c} = \bigcup_{\alpha < \mathfrak{c}} \{z_n^\alpha : n < \omega\}$. All the z_n^α 's are distinct and isolated in $S_\mathfrak{c}$. If $f \in \omega^\mathfrak{c}$ then

$$V(f) = \{0\} \cup \bigcup_{\alpha < \mathfrak{c}} \{z_n^\alpha : n \geq f(\alpha)\}$$

is a basic open neighborhood of 0 in $S_\mathfrak{c}$. Observe that the sequential fan $S_\mathfrak{c}$ is nothing but the space obtained by identifying the limit points of the topological sum of continuum many convergent sequences.

If $F \subseteq \mathfrak{c}$ then we put $S_F = \{0\} \cup \bigcup_{\alpha \in F} \{z_n^\alpha : n < \omega\}$. This notation will be used without explicit reference later.

* Corresponding author. E-mail: vanmill@cs.vu.nl.

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A space X has *countable tightness* if whenever $A \subseteq X$ and $x \in \overline{A}$, there exists a countable set $B \subseteq A$ such that $x \in \overline{B}$. If X and Y are spaces with $X \subseteq Y$ then X has countable tightness in Y if whenever $A \subseteq Y$ and $x \in \overline{A} \cap X$, there exists a countable set $B \subseteq A$ such that $x \in \overline{B}$.

A space X has *countable fan-tightness* if for any countable family $\{A_n: n \in \omega\}$ of subsets of X satisfying $x \in \bigcap_{n \in \omega} \overline{A_n}$ it is possible to select finite sets $K_n \subseteq A_n$ in such a way that $x \in \overline{\bigcup_{n \in \omega} K_n}$.

Let X be a space and let $p \in X$. We say that a family of subsets \mathcal{E} of X *clusters at* p if for every neighborhood U of p there exists $E \in \mathcal{E}$ such that $|E \cap U| \geq \omega$. We say that p is a *tight point* of X if for every family \mathcal{E} of subsets of X that clusters at p there exists a *countable* subfamily \mathcal{F} of \mathcal{E} that clusters at p . Observe that if p is an isolated point of X then p is tight for trivial reasons.

2. Tight points

Let X be a space. If X has countable fan-tightness then X has countable tightness. But not conversely. The space S_c has countable tightness, but does not have countable fan-tightness.

Proposition 2.1. *Let X be a space. If every point of X is tight then X has countable fan-tightness.*

Proof. Fix $p \in X$ and subsets A_n of X for $n < \omega$ such that $p \in \bigcap_{n < \omega} \overline{A_n}$. We assume without loss of generality that for every n , A_n is infinite and that $p \notin A_n$. Let \mathcal{S} be the collection of all subsets $S = \{x_n: n < \omega\}$ of X such that for every n , $x_n \in A_n$. We claim that \mathcal{S} clusters at p . To see this, let U be an arbitrary neighborhood of p . Since $U \cap A_n$ is infinite for every n , we may pick points x_n in X such that for every n ,

$$x_n \in (U \cap A_n) \setminus \{x_0, \dots, x_{n-1}\}.$$

Then $S = \{x_n: n < \omega\}$ belongs to \mathcal{S} , is infinite and is contained in U .

Since p is a tight point of X , there are $S_k = \{x_n^k: n < \omega\} \in \mathcal{S}$ for $k < \omega$ such that $\{S_k: k < \omega\}$ clusters at p . For every n , define

$$K_n = \{x_n^0, x_n^1, \dots, x_n^n\}$$

and $K = \bigcup_{n < \omega} K_n$. Let U be an arbitrary neighborhood of p . Then for some k we have that $U \cap S_k$ is infinite. Since $S_k \setminus K$ is finite we conclude that U intersects K . This shows that $p \in \overline{K}$, which is as required. \square

Let X be a space. If every point of X is tight, then X has countable fan-tightness as was shown in Proposition 2.1. But the converse is not true (at least, consistently). If $p \in \beta\omega \setminus \omega$ is a P-point then $\omega \cup \{p\}$ has countable fan-tightness (see [2, Proposition 2]). But p is not a tight point of $\omega \cup \{p\}$ by our next result.

Proposition 2.2. *Let $p \in \beta\omega \setminus \omega$. Then p is not a tight point of $\omega \cup \{p\}$.*

Proof. Suppose first that p is a P-point and put $\mathcal{S} = \{\omega \setminus P : P \in p\}$. We claim that \mathcal{S} clusters at p . To see this, let $P \in p$ and split P into two infinite sets, say P_0 and P_1 . Without loss of generality, assume that $P_0 \in p$. Then $\omega \setminus P_0 \in \mathcal{S}$ and $|(\omega \setminus P_0) \cap P| = \omega$.

Now suppose that \mathcal{B} is a countable subfamily of \mathcal{S} that clusters at p . Then $\{\omega \setminus B : B \in \mathcal{B}\}$ is a countable subfamily of p . Since p is a P-point, there exists $P \in p$ such that for all $B \in \mathcal{B}$ we have $|P \setminus (\omega \setminus B)| < \omega$. But then $P \cap B$ is finite for every $B \in \mathcal{B}$, which is a contradiction.

If p is not a P-point, then there is a partition $\{A_n : n < \omega\}$ of ω into infinite sets such that for every n , $A_n \notin p$, and if $P \in p$ then $|P \cap A_n| = \omega$ for infinitely many n . Now put

$$\mathcal{S} = \left\{ \bigcup_{n < \omega} F_n : F_n \subseteq A_n \text{ finite, } n < \omega \right\}.$$

It is clear that \mathcal{S} clusters at p . Assume that some countable subfamily \mathcal{T} of \mathcal{S} clusters at p . Let $\mathcal{T} = \{T_n : n < \omega\}$ and for every $n < \omega$, let $T_n = \bigcup_{m < \omega} F_m^n$, with F_m^n a finite subset of A_m for every m . Put

$$K = F_0^0 \cup (F_1^0 \cup F_1^1) \cup (F_2^0 \cup F_2^1 \cup F_2^2) \cup \dots.$$

Then $K \cap A_n$ is finite for every n , and as a consequence, $U = \omega \setminus K \in p$. But $|U \cap T_n| < \omega$ for every n , which is a contradiction. \square

We now show that the points of a regular countably compact spaces of countable tightness are tight.

Theorem 2.3. *Let X be a subspace of a regular countably compact space Y . If X has countable tightness in Y then every point of X is tight.*

Proof. Let \mathcal{E} be a family of subsets of X that clusters at a given point $p \in X$. For every neighborhood U of p in Y first pick a neighborhood $V(U)$ of p in Y such that $\overline{V(U)} \subseteq U$ (here we use the fact that Y is regular). Next pick an element $E(U) \in \mathcal{E}$ such that $F(U) = E(U) \cap V(U)$ is infinite. Passing to a subset of $F(U)$ if necessary, we may assume that $F(U)$ is countably infinite. Since Y is countably compact, we may pick for every U an accumulation point $p(U)$ of $F(U)$ in Y . Put $A = \{p(U) : U \text{ is a neighborhood of } p \text{ in } Y\}$. Then $p \in \overline{A}$ and since X has countable tightness in Y , there is a countable family of neighborhoods \mathcal{V} of p such that

$$p \in \overline{\{p(V) : V \in \mathcal{V}\}}.$$

We claim that the family $\{E(V) : V \in \mathcal{V}\}$ clusters at p . To this end, let U be a neighborhood of p in X and let U' in Y be open such that $U' \cap X = U$. There exists $V \in \mathcal{V}$ such that $p(V) \in U'$. Since U' is a neighborhood of $p(V)$ and $p(V)$ is a cluster point of $F(V)$ we have that $U \cap F(V) = U' \cap F(V)$ is infinite. Since $F(V) \subseteq E(V)$ this proves that $U \cap E(V)$ is infinite, which is as required. \square

The following result is partly due to Arhangel'skiĭ and Bella [2].

Corollary 2.4. *Let X be a regular countably compact space. Then X has countable tightness if and only if X has countable fan-tightness if and only if every point of X is tight.*

Proof. If X has countable tightness, then every point of X is tight by Theorem 2.3 which in turn implies that X has countable fan-tightness by Proposition 2.1. \square

The following result of which we present a new proof (for another proof, see Arhangel'skiĭ and Bella [2]) is due to Malykhin [3].

Corollary 2.5. *If $p \in \beta\omega \setminus \omega$ then $\omega \cup \{p\}$ cannot be embedded in a regular countably compact space with countable tightness.*

Proof. Striving for a contradiction, assume that X is a regular countably compact space with countable tightness which contains $\omega \cup \{p\}$ as a subspace. Then every point of X is tight by Corollary 2.5 and so p is a tight point of $\omega \cup \{p\}$ which contradicts Proposition 2.2. \square

This result can be improved, as we will show in the remaining part of this section.

Proposition 2.6. *Let X be a regular space with countable fan-tightness. If all closed separable subspaces of X are Lindelöf with points G_δ then for any $A \subseteq X$ and any $p \in \bar{A} \setminus A$ there exists a countable set $B \subseteq A$ such that p is the only accumulation point of B .*

Proof. Let $A \subseteq X$ and $p \in \bar{A} \setminus A$. Since X has countable tightness, there exists a countable set $C \subseteq A$ such that $p \in \bar{C}$. For any $x \in \bar{C} \setminus \{p\}$ fix an open set U_x satisfying $x \in U_x$ and $p \notin \bar{U}_x$ and let \mathcal{U} be the family so obtained. Since \bar{C} has the Lindelöf property and p is a G_δ point in \bar{C} , it follows that even the subspace $\bar{C} \setminus \{p\}$ has the Lindelöf property. So \mathcal{U} has an open countable refinement $\{V_n: n < \omega\}$ such that the family $\{V_n \cap C: n < \omega\}$ is locally finite in $\bar{C} \setminus \{p\}$. For any n we have $p \in \overline{\bigcup\{V_m \cap C: m \geq n\}}$ and therefore we can select a finite set $K_n \subseteq \bigcup\{V_m \cap C: m \geq n\}$ such that $p \in \overline{\bigcup\{K_n: n < \omega\}}$. Putting $B = \bigcup\{K_n: n < \omega\}$ we get what we want. \square

Recall that, given a Tychonoff space X , a point $p \in \beta X \setminus X$ is said to be *far* if it is not in the closure of any closed discrete subset of X . If $p \in \beta X \setminus X$ is not far then it is called *near*.

Corollary 2.7. *Let X be a Tychonoff space whose closed separable subspaces have the Lindelöf property and $p \in \beta X \setminus X$. If $X \cup \{p\}$ has countable fan-tightness then p is near.*

Malykhin's result quoted above can now be generalized as follows:

Theorem 2.8. *If X is a metrizable space and $p \in \beta X \setminus X$ then $X \cup \{p\}$ cannot be embedded into a countably compact regular space with countable tightness.*

Proof. Assuming the contrary, the space $X \cup \{p\}$ must have countable fan-tightness. Consequently, as stated in Corollary 2.7, there exists a countable closed discrete set $B \subseteq X$ for which $p \in \overline{B}$. But then, also $B \cup \{p\}$ can be embedded into a countably compact regular space with countable tightness—in contrast with Corollary 2.5. \square

Recall that a space X is said to be *bisequential* provided that for every filter ξ and every point p in the adherence of ξ there exists a filter ν with a countable base which converges to p and is synchronous with ξ . *Synchronous* means that for each $A \in \xi$ and each $B \in \nu$ the intersection $A \cap B$ is not empty.

Arhangel'skiĭ (see [1]) has shown that the product of a bisequential space with any space of countable tightness has still countable tightness. By Theorem 3.1 below, it then follows that each point of a bisequential space is tight. Here is a direct and easy proof of this assertion.

Proposition 2.9. *Every point of a bisequential space is tight.*

Proof. Let X be a bisequential space and \mathcal{E} a collection of subsets of X which clusters at p . Define $\xi = \{\bigcup_{E \in \mathcal{E}} E \setminus F : F \subseteq E \text{ finite}\}$. It is clear that ξ is a prefilter and p is in the adherence of ξ . Let \mathcal{U} be a countable base of a filter ν converging to p and which is synchronous with ξ . Since ν and ξ are synchronous, for every $U \in \mathcal{U}$ there must exist some $E(U) \in \mathcal{E}$ such that $|U \cap E(U)| \geq \omega$. The fact that ν converges to p implies that every neighbourhood of p contains some $U \in \mathcal{U}$ and consequently the subcollection $\{E(U) : U \in \mathcal{U}\}$ clusters at p . \square

3. The main result

We now present our main result.

Theorem 3.1. *Let X be a countable space. Then $X \times S_c$ has countable tightness if and only if every point of X is tight.*

Proof. Assume that $X \times S_c$ has countable tightness and suppose that the family \mathcal{E} clusters at $p \in X$. We may assume without loss of generality that every element $E \in \mathcal{E}$ is countably infinite and does not contain p . List \mathcal{E} as $\{E_\alpha : \alpha < \mathfrak{c}\}$ (repetitions are permitted) and E_α as $\{e_n^\alpha : n < \omega\}$ (repetitions are NOT permitted). Let

$$A = \bigcup_{\alpha < \mathfrak{c}} \{\langle e_n^\alpha, z_n^\alpha \rangle : n < \omega\}.$$

We claim that $\langle p, 0 \rangle \in \overline{A}$. To see this, let U and $V(f)$ be arbitrary neighborhoods of p in X and $0 \in S_c$, respectively. There exists $\alpha < \mathfrak{c}$ such that $|U \cap E_\alpha| = \omega$. Pick $n < \omega$ so large that $e_n^\alpha \in U \cap E_\alpha$ and $n \geq f(\alpha)$. Then

$$\langle e_n^\alpha, z_n^\alpha \rangle \in (U \times V(f)) \cap A,$$

as required.

Since $X \times S_c$ has countable tightness, there is a countable subset F of \mathfrak{c} such that if

$$B = \bigcup_{\alpha \in F} \{ \langle e_n^\alpha, z_n^\alpha \rangle : n < \omega \}$$

then $\langle p, 0 \rangle \in \overline{B}$. Now put $\mathcal{F} = \{E_\alpha : \alpha \in F\}$. We claim that \mathcal{F} clusters at p . To see this, let U be an arbitrary neighborhood of p in X . Striving for a contradiction, assume that for every $\alpha \in F$ we have $|U \cap E_\alpha| < \omega$. For every $\alpha \in F$ pick $n_\alpha < \omega$ such that if $n \geq n_\alpha$ then $e_n^\alpha \notin U$. Define $f \in \omega^\mathfrak{c}$ as follows: $f(\alpha) = n_\alpha$ if $\alpha \in F$ and $f(\alpha) = 0$ otherwise. Pick $\alpha \in F$ and $n < \omega$ such that $\langle e_n^\alpha, z_n^\alpha \rangle \in U \times V(f)$. Then $e_n^\alpha \in U \cap E_\alpha$ and so $n < n_\alpha$. However, $z_n^\alpha \in V(f)$ which implies $n \geq f(\alpha) = n_\alpha$. This is a contradiction.

Now assume that p is a tight point of X . We will prove that the tightness of $X \times S_c$ at $\langle p, 0 \rangle$ is countable. Since X is countable and every point of S_c other than 0 is isolated, this clearly suffices to prove that $X \times S_c$ has countable tightness.

Let $A \subseteq (X \times S_c) \setminus \{ \langle p, 0 \rangle \}$ be such that $\langle p, 0 \rangle \in \overline{A}$. Put

$$B = \{x \in X : \langle x, 0 \rangle \in \overline{(\{x\} \times S_c) \cap A}\}.$$

Assume first that $p \in \overline{B}$. Since S_c has countable tightness, for every $x \in B$ there exists a countable $S_x \subseteq (\{x\} \times S_c) \cap A$ with $\langle x, 0 \rangle \in \overline{S_x}$. Then $\langle p, 0 \rangle \in \bigcup_{x \in B} \overline{S_x}$ and so we are done.

So we may assume without loss of generality that for every $x \in X$ we have

$$\langle x, 0 \rangle \notin \overline{(\{x\} \times S_c) \cap A}.$$

Now let U be an arbitrary neighborhood of p in X . We claim that there exists $\alpha(U) < \mathfrak{c}$ such that

$$\{n < \omega : (\exists x \in U)(\langle x, z_n^{\alpha(U)} \rangle \in A)\}$$

is infinite. If not, then for every $\alpha < \mathfrak{c}$ pick $n_\alpha < \omega$ such that if $n \geq n_\alpha$ and $x \in U$ then $\langle x, z_n^\alpha \rangle \notin A$. Put $f(\alpha) = n_\alpha$ for every α . Then $(U \times V(f)) \cap A = \emptyset$, which is a contradiction. This proves the claim.

Pick an arbitrary $x \in U$ and observe that

$$\langle x, 0 \rangle \notin \overline{(\{x\} \times S_c) \cap A}.$$

Since $\langle x, z_n^{\alpha(U)} \rangle \rightarrow \langle x, 0 \rangle$ ($n \rightarrow \infty$), this implies that there are only finitely many n for which $\langle x, z_n^{\alpha(U)} \rangle \in A$. So by the above we may pick for every $n < \omega$ an element $x_n^{\alpha(U)} \in U$ and an integer $m(n)$ such that

- (i) if $n \neq m$ then $x_n^{\alpha(U)} \neq x_m^{\alpha(U)}$;
- (ii) $\langle x_n^{\alpha(U)}, z_{m(n)}^{\alpha(U)} \rangle \in A$ for every n ;

(iii) $m(n) \rightarrow \infty$ ($n \rightarrow \infty$).

Put $E(U) = \{x_n^{\alpha(U)} : n < \omega\}$.

The family $\{E(U) : U \text{ is a neighborhood of } p\}$ clusters at p . There consequently is a countable family \mathcal{U} of neighborhoods of p such that $\mathcal{F} = \{E(U) : U \in \mathcal{U}\}$ clusters at p . Put $F = \{\alpha(U) : U \in \mathcal{U}\}$ and

$$B = (X \times S_F) \cap A,$$

respectively. Then B is countable and we claim that $\langle p, 0 \rangle \in \overline{B}$. To see this, let V and $V(f)$ be arbitrary neighborhoods of p in X and 0 in S_c , respectively. There exists $U \in \mathcal{U}$ such that $|E(U) \cap V| = \omega$. By (iii) we may pick $n < \omega$ so large that

$$x_n^{\alpha(U)} \in E(U) \cap V \quad \text{and} \quad m(n) \geq f(\alpha(U)).$$

We conclude that

$$\langle x_n^{\alpha(U)}, z_{m(n)}^{\alpha(U)} \rangle \in A \cap (V \times V(f)),$$

which is as required. \square

The following result answers a question in [2] in the affirmative.

Corollary 3.2. *Let X be a space. If $X \times S_c$ has countable tightness then X has countable fan-tightness.*

Proof. Suppose that $X \times S_c$ has countable tightness. Then X has countable tightness, and so it suffices to prove that every countable subspace of X has countable fan-tightness. So without loss of generality, assume that X is countable. By Theorem 3.1, every point of X is tight and so X has countable fan-tightness by Proposition 2.1. \square

As mentioned in Section 2, in [2] it is shown that every countably compact regular space with countable tightness has countable fan-tightness. Corollary 3.2 can be used to present yet another proof of this fact. Indeed, it suffices to take into account the well-known fact that the product of a countably compact regular space of countable tightness with a sequential space has still countable tightness.

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